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Adaptive Mittag-Leffler stabilization of commensurate fractional-order nonlinear systems

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Abstract— **In this paper, a new adaptive control technique called adaptive fractional-order backstepping is proposed, for a class of commensurate fractional-order nonlinear systems with uncertain constant parameters. Using the adaptive fractional-order backstepping as a basic design tool, we show how to explicitly construct an adaptive feedback control laws that solve the Mittag-Leffler stabilization problem of uncertain commensurate fractional-order nonlinear systems. The global convergence of the closed-loop systems is guaranteed in the sense of Mittag-Leffler stability. The efficiency of the proposed technique is demonstrated in simulation finally.**

I. INTRODUCTION

Currently, fractional-order systems have been widely used in the area of dynamical systems and control. This is mainly due to that many physical systems are well characterized by fractional-order differential equations [1]–[3], such as ultracapacitor, viscoelastic mechanical system and diffusive system, etc. For more details on this area, one can refer to the monographs [1]–[4], the papers [5]–[20], [22]–[28] and the reference therein.

Uncertainties are common phenomena in all dynamical systems including fractional-order systems. Recently, many results are devoted to cope with uncertainties in fractional-order systems. By use of linear matrix inequality (LMI) technique, several works concentrated on robust stabilization of uncertain fractional-order systems. The LMI-based robust stabilization conditions of uncertain fractional-order linear systems were proposed in [5]–[7] early. For some recent works, one can refer to [8]–[10] and the references therein. Besides, $\mathscr{H}_\infty$ control problems of fractional-order systems were investigated in [11]–[13] recently. For some noises that appear in processes or in measurements, [14] proposed self-tuning control for stochastic fractional-order systems. However, initially uncertain (or time-varying) parameters always exist in fractional-order systems. Adaptive control was famous to deal with them, which has been introduced for fractional-order systems. By extending the classical model reference adaptive control (MRAC), fractional-order reference model or adaptive laws shown in [15]–[18] and the references therein. Another important method is fractional-order sliding mode control, which were presented in [19], [20] and the references therein for uncertain fractional-order linear or nonlinear systems.

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As we know, adaptive backstepping is a well-known technique of stabilizing uncertain nonlinear systems with unknown constant parameters [21]. However, to the authors' best knowledge, this technique is restricted to the classical uncertain integer-order nonlinear systems. There few results on uncertain fractional-order nonlinear systems stabilization via adaptive backstepping. Motivated by the mentioned developments, we devote to solve the Mittag-Leffler stabilization problem of uncertain fractional-order nonlinear systems via adaptive backstepping. As the resulting adaptive control laws are with fractional-order forms, we call such methodology the adaptive fractional-order backstepping.

An attention should be paid to, Mittag-Leffler stability just describes the pseudo-state trajectories, not the real states of fractional-order systems, which you can refer to references [22]–[24] for distinguishing them. So, we call the Lyapunov function construction for Mittag-Leffler stability as fractional Lyapunov function, which cannot be seen as energy-like functions as asserted in [24], [25].

In our contributions, the Mittag-Leffler stabilization problem of uncertain commensurate fractional-order nonlinear systems is solved with a guaranteed global convergence of closed-loop systems. Firstly a general framework of Lyapunov-like based design is well defined via adaptive control fractional Lyapunov function (acflf) for fractional-order systems. Within this framework, adaptive fractional-order backstepping is proposed by extending the classical adaptive backstepping for uncertain fractional-order nonlinear systems. The analytic forms of adaptive feedback control laws are designed via the adaptive fractional-order backstepping. The proposed technique is verified finally in an application.

The paper is organized as follows. In Section II, some definitions and the concept of acflf are introduced. In Section III, fractional-order backstepping is reviewed firstly. By extending it, the adaptive fractional-order backstepping is concluded by stabilizing uncertain fractional-order nonlinear systems. The analytic forms of adaptive feedback control laws are derived. The theoretical result is verified in Section IV. Finally, we conclude the paper in Section V.

II. PRELIMINARIES

The Caputo fractional-order derivative is used.

Definition 2.1: [4] Let $f(t)$ is a real continuously differentiable function. The Caputo fractional-order derivative with order $0<\nu<1$ on $t>0$ is defined by

$$D_t^\nu f(t) = \frac{1}{\Gamma(n-\nu)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\nu-n+1}} d\tau,$$

where $n = \lceil \nu \rceil, \nu > 0$, $\lceil \cdot \rceil$ is the ceiling function.

The fractional-order derivative of a constant C is 0. For simplicity, the symbol D^ν is shorted for D_t^ν, where t is the time.

Definition 2.2: [26] A continuous function $\gamma : [0,t) \to [0,\infty)$ is said to be the K-class function if it is strictly increasing and $\gamma(0) = 0$.

Theorem 2.1: [26] Let $x(t) = 0$ be the equilibrium point of the fractional-order system $D^\nu x = f(x,t), x \in \Omega$, where Ω is a neighborhood region of origin. Assume there exists a Lyapunov function $V(t,x(t)) : [0,\infty] \times \mathbb{R}^n \to \mathbb{R}$ and K-class functions $\gamma_i, i = 1,2,3$ satisfying

$$\begin{aligned}&(\mathrm{i})\gamma_1(\|x\|) \le V(t,x(t)) \le \gamma_2(\|x\|),\\&(\mathrm{ii})D^\nu V(t,x(t)) \le -\gamma_3(\|x\|).\end{aligned}$$

Then the system is asymptotically Mittag-Leffler stable. Moreover, if $\Omega = \mathbb{R}^n$, the system is globally asymptotically Mittag-Leffler stable.

Remark 2.1: Theorem 2.1 deals with pseudo-states of fractional-order systems and tells us the sufficient conditions of Lyapunov functions to make them Mittag-Leffler stable. Actually, as illustrated in [23]–[25], this kind Lyapunov functions are not accurate energy descriptions for fractional-order systems, so we call them fractional Lyapunov functions.

To construct fractional Lyapunov functions for fractional-order systems, the power law for fractional-order derivative is introduced now.

Lemma 2.1: Let $x(t) \in \mathbb{R}$ be a real continuously differentiable function. Then for any $p = 2^n, n \in \mathbb{N}, D^\nu x^p(t) \le px^{(p-1)}(t)D^\nu x(t)$, where $0 < \nu < 1$ is the fractional order.

Proof: A simple case of $p = 2$ was shown by [28]. For the proof of this general version one can see [27]. ■

Corollary 2.1: Let $x(t) \in \mathbb{R}$ be a real continuously differentiable function. Then for $p = 2, \frac{1}{2}D^\nu x^2(t) \le x(t)D^\nu x(t)$, where $0 < \nu < 1$ is the fractional order.

Corollary 2.2: Let $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ be a real continuous and differentiable vector function. Then $D^\nu[x(t)^T Px(t)] \le 2x(t)^T PD^\nu x(t)$, where $0 < \nu < 1$ is the fractional order and $P = diag[p_1, \ldots, p_n] > 0$.

It will be demonstrated in Section III that $\frac{1}{2}x(t)^T Px(t)$ (or $P = I$) is always a reasonable fractional Lyapunov function. Next, the concept of acflf is introduced to test whether an uncertain fractional-order system is feedback Mittag-Leffler stabilizable.

Definition 2.3: A smooth function $V(t,x(t),\tilde{\theta}) : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m$ is called a acflf for $D^\nu x(t) = f(x,u,\theta), x \in \mathbb{R}^n, f(0,0,\cdot) = 0$ with the adaptive control law $u = \alpha(x,\hat{\theta})$ if there exist three K-class functions $\gamma_i, i = 1,2,3$, such that

$$\begin{aligned}&(\mathrm{i})\gamma_1(\|\bar{x}\|) \le V(t,x(t),\|\tilde{\theta}\|) \le \gamma_2(\|\bar{x}\|),\\&(\mathrm{ii})D^\nu V(t,x(t),\|\tilde{\theta}\|) \le -\gamma_3(\|\bar{x}\|).\end{aligned}$$

where $\bar{x} = [x^\top, \hat{\theta}^\top]^\top, \theta \in \mathbb{R}^m$ is the unknown parameter, the parameter estimate error is $\tilde{\theta} = \theta - \hat{\theta}$ and $D^\nu \hat{\theta} = \tau(x,\hat{\theta})$ is the adaptive law of the parameter estimate.

The symbol $\|\cdot\|$ represents Euclidean norm in the following design, which will not be declared.

When the adaptive parameters appear in fractional Lyapunov function V, this V is called adaptive control fractional Lyapunov function (acflf). With the known fractional order ν, the aim of adaptive Mittag-Leffler stabilization of uncertain fractional-order nonlinear systems is to design an adaptive feedback control law $u = \alpha(x,\tilde{\theta}), D^\nu \hat{\theta} = \tau(x,\hat{\theta})$ such that the closed-loop systems is (globally) asymptotically Mittag-Leffler stable. Actually finding α, τ and V satisfying (i) and (ii) in Definition 2.3 is a difficult task in most cases.

Remark 2.2: As the resulting controllers from Lemma 2.1 and its corollaries have the same right formula as the classical forms [21], so they are conservative. These conditions are sufficient for Mittag-Leffler stabilizability of uncertain fractional-order nonlinear systems. Besides, it is possible that there exist other better candidate fractional Lyapunov functions, which may contradict with Theorem 2.1. However, they are always valid for some specific fractional-order nonlinear systems.

III. MAIN RESULTS

A common form of nonlinearities appears multiplied with physical constants, often poorly known or dependent on the slowly changing environment. We consider the unknown constant parameters appear linearly in the fractional-order models. In presence of such parametric uncertainties, the adaptive fractional-order backstepping is introduced to achieve convergence of the closed-loop system.

Before the adaptive fractional-order backstepping, we review fractional-order backstepping in an example. For the details one can refer to [2], [27].

Example 3.1: Consider a fractional-order nonlinear planar system

$$\begin{cases} D^\nu x = x\xi + x\theta \\ D^\nu \xi = u \end{cases},$$

where $x, \xi \in \mathbb{R}$ are the states and $u \in \mathbb{R}$ is the control input. θ is the unknown bounded parameter, but we do not know its bound.

The static feedback controller is considered here via fractional-order backstepping. Let 1st fractional Lyapunov function $V=\frac{1}{2}x^2, z=\xi-\alpha(x)$, and ξ viewed as the virtual control, we have

$$D^\nu V \le xD^\nu x = x^2[z+\alpha(x)+\theta].$$

Choose $\alpha=-Cx^2, C>0$, the fractional-order system becomes

$$\begin{cases} D^\nu x = xz - Cx^3 + x\theta \\ D^\nu z = u - D^\nu \alpha(x) \end{cases}.$$

Choose the candidate acflf $V_a(x,\xi)=V(x)+\frac{1}{2}[\xi-\alpha(x)]^2$, we have

$$D^\nu V_a(x,\xi) \le -Cx^4 + z[u+x^2-D^\nu\alpha(x)]+x^2\theta.$$

If the control is chosen as $u=-C_1[\xi-\alpha(x)]-x^2+D^\nu\alpha(x), C_1>0$, we have

$$D^\nu V_a(x,\xi) \le -Cx^4 - C_1 z^2 + x^2\theta.$$

where $C, C_1 > 0$ are constants. It is obvious that the global boundedness can be guaranteed by choosing $C > \|\theta\|_\infty$.

In Example 3.1, the static controller can guarantee that in the presence of uncertain bounded uncertainties the closed-loop states remain bounded (semi-global stability), the feedback gain may increase too large. If the uncertain parameters in fractional-order nonlinear models are unknown, the fractional-order backstepping may be invalid. Therefore adaptive fractional-order backstepping need to be introduced.

We have the following assumption firstly.

Assumption 3.1: Let the uncertain fractional-order nonlinear system

$$D^\nu x = f(x)+F(x)\theta+g(x)u,$$

where $x\in\mathbb{R}^n$ is the pseudo-state, $\theta\in\mathbb{R}^m$ is the unknown constant parameter and $u\in\mathbb{R}$ is the control input. There exists a adaptive feedback control law $u=\alpha(x,\hat{\theta})$ and a K-class function γ such that

$$x^\top[f(x)+F(x)\hat{\theta}+g(x)\alpha(x)] \le -\gamma(\|\bar{x}\|), \qquad (1)$$

$$D^\nu\hat{\theta} = \Gamma F(x)^\top x, \qquad (2)$$

where $\bar{x}=[x^\top,\hat{\theta}^\top]^\top, \hat{\theta}\in\mathbb{R}$ is the parameter estimate and $\Gamma=diag[p_1,\dots,p_n]>0$ is the gain matrix of the adaptive law.

Denote $\tilde{\theta}=\theta-\hat{\theta}$, let the candidate aclf

$$V_a=\frac{1}{2}x^\top x+\frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}.$$

The sufficiency of (1) and (2) is obvious.

For the scalar system $x,\theta\in\mathbb{R}$, Assumption 3.1 is always valid for $g(x)\neq 0, x\in\mathbb{R}$. In this case, the adaptive control u can be set to $\alpha(x,\hat{\theta})=-\frac{1}{g(x)}[f(x)+F(x)\hat{\theta}+Cx]$, and $D^\nu\hat{\theta}=F(x)x$. With this adaptive control law, we have $D^\nu V_a \le -C\|x\|$, where $C>0$ is a constant. It is obvious that unless $x=0$, we have $D^\nu V_a<0$. There exists a K-class function γ such that $D^\nu V_a \le -\gamma(\|\bar{x}\|)$.

A special attention should be paid to the conservative of possible control laws, which satisfies (1) and (2), because the choice of acflf is conservative. However, the efficiency is obvious for common uncertain fractional-order nonlinear systems, which will be justified later.

Theorem 3.1: Let the uncertain fractional-order nonlinear system

$$\begin{cases} D^\nu x = f(x)+F(x)\theta+g(x)\xi \\ D^\nu\xi = u \end{cases},$$

where $x\in\mathbb{R}^n, \xi\in\mathbb{R}$ is the pseudo-states, $\theta\in\mathbb{R}^m$ is an unknown constant and $u\in\mathbb{R}$ is the control input. Let $D^\nu x = f(x)+F(x)\theta+g(x)\xi$ satisfies Assumption 3.1 with $\xi\in\mathbb{R}$ as its virtual control. If the acflf is taken by

$$V_a(z_1,z_2,\tilde{\theta})=\frac{1}{2}z_1^\top z_1+\frac{1}{2}z_2^2+\frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}, \qquad (3)$$

where $z_1=x, z_2=\xi-\alpha(x,\hat{\theta})$ and $\tilde{\theta}=\theta-\hat{\theta}$ is the parameter estimate error, that is, there exists an adaptive feedback control u which renders the equilibrium of $(z_1,z_2,\tilde{\theta})$ globally asymptotically Mittag-Leffler stable. The feedback control law can be chosen by

$$u=-x^\top g(x)-C_1[\xi-\alpha(x,\hat{\theta})]+D^\nu\alpha, \qquad (4)$$

$$D^\nu\hat{\theta}=\Gamma F(x)^\top x, \qquad (5)$$

where the adaptive parameter $\hat{\theta}$ is updated by (5), and $\Gamma=diag[p_1,\dots,p_n]>0$ is the gain matrix of the adaptive law.

Proof: Two steps in this proof are presented.

Step 1. Let $z_1=x$ and ξ viewed as the virtual control, the error $z_2=\xi-\alpha(x,\hat{\theta})$, we have

$$D^\nu z_1 = f(z_1)+F(z_1)\theta+g(z_1)[z_2+\alpha(x,\hat{\theta})].$$

Note $\tilde{\theta}=\theta-\hat{\theta}$, the 1st fractional Lyapunov function $V_1(z_1,\tilde{\theta})=\frac{1}{2}z_1{}^\top z_1+\frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}$. With Assumption 3.1, we have

$$\begin{aligned} D^\nu V_1 \le & -\gamma(\|\bar{x}\|)+z_1^\top g(z_1)z_2 \\ & +\tilde{\theta}^\top[F(z_1)^\top z_1-\Gamma^{-1}D^\nu\hat{\theta}]. \end{aligned}$$

We postpone the choice of update law for $\hat{\theta}$ until the next step.

Step 2. To design the adaptive control u for $D^\nu z_2 = u - D^\nu\alpha$. Consider the aclf (3), we have

$$\begin{aligned} D^\nu V_a \le & -\gamma(\|\bar{x}\|)+\tilde{\theta}^\top[F(z_1)^\top z_1-\Gamma^{-1}D^\nu\hat{\theta}] \\ & +z_2[z_1^\top g(z_1)+u-D^\nu\alpha]. \end{aligned}$$

One control and the adaptive law can be chosen by (4) and (5) respectively. With Assumption 3.1, we have

$$D^{v}V_a \leq -\gamma(\|\bar{x}\|) - C_1 z_2^2.$$

By use of Euclidean norm, there exists a K-class function $\bar{\gamma}(\|\bar{\bar{x}}\|) = \gamma(\|\bar{x}\|) + C_1 z_2^2, \bar{\bar{x}} = [\bar{x}^\top, z_2]^\top$.

With respect to Theorem 2.1, the acflf holds globally. So far, the proof is completed. ∎

Now, we reconsider Example 3.1 using the above adaptive fractional-order backstepping scheme.

Example 3.2: (Continued Example 3.1) At first, let $z_1 = x$ and ξ viewed as the virtual control, the error $z_2 = \xi - \alpha(x, \hat{\theta})$, we have

$$D^{v}z_1 = z_1[z_2 + \alpha(x, \hat{\theta})] + z_1\theta.$$

Note $\tilde{\theta} = \theta - \hat{\theta}$, let the 1st fractional Lyapunov function $V_1(z_1, \hat{\theta}) = \frac{1}{2}z_1^2 + \frac{1}{2\rho}\tilde{\theta}^2, \rho > 0$. If choose $\alpha(x, \hat{\theta}) = -C_1 - \hat{\theta}$, we have

$$D^{v}V_1 \leq -C_1 z_1^2 + z_1^2 z_2 + \tilde{\theta}[z_1^2 - \frac{1}{\rho}D^{v}\hat{\theta}].$$

The to design the adaptive control u with the acflf (3), we have $D_a^{v} \leq -C_1 z_1^2 + \tilde{\theta}[z_1^2 - \frac{1}{\rho}D^{v}\hat{\theta}]$.

One control and the adaptive law can be chosen by

$$u = -C_2 z_2 - z_1^2 + D^{v}\alpha, D^{v}\hat{\theta} = \rho z_1^2.$$

Thus, we have $D^{v}V_a \leq -C_1 z_1^2 - C_2 z_2^2$.

According to Theorem 2.1, the closed-loop system is globally asymptotically stable.

The adaptive fractional-order backstepping is shown in Theorem 3.1 with respect to a single uncertain parameter. The following theorem concerns a general form.

Theorem 3.2: Let the parametric strict-feedback form of uncertain fractional-order nonlinear system

$$\begin{cases} D^{v}x_1 = x_2 + \varphi_1^\top(x_1)\theta \\ D^{v}x_2 = x_3 + \varphi_2^\top(x_1, x_2)\theta \\ \vdots \\ D^{v}x_{n-1} = x_n + \varphi_{n-1}^\top(x_1, \ldots, x_{n-1})\theta \\ D^{v}x_n = \beta(x)u + \varphi_n^\top(x)\theta \end{cases},$$

where $\beta(x) \neq 0$ for all $x \in \mathbb{R}^n, \theta \in \mathbb{R}^m$ is an unknown constant and $u \in \mathbb{R}$ is the control input. If the acflf is taken by

$$V_a(z_1, \ldots, z_n, \hat{\theta}) = \frac{1}{2}\sum_{i=1}^{n} z_i^2 + \frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}, \tag{6}$$

where $z_1 = x_1, z_i = x_i - \alpha_{i-1}(z_1, \ldots, z_{i-1}, \hat{\theta}), i = 2, \ldots, n$ and $\tilde{\theta} = \theta - \hat{\theta}$ is the parameter estimate error, that is, there exists an adaptive feedback control u which renders the closed-loop system globally asymptotically Mittag-Leffler stable on Λ, where $\Lambda = \{(z_1, \ldots, z_n, \tilde{\theta}) | z \neq 0\}$. And the boundedness of the closed-loop systems is guaranteed on $\mathbb{R}^{m+n} \backslash \Lambda$. The adaptive feedback control law can be chosen by

$$u = -\frac{1}{\beta(x)}[C_n z_n + z_{n-1} - \varphi_n^\top(x)\hat{\theta} + D^{v}\alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta})], \tag{7}$$

$$D^{v}\hat{\theta} = \Gamma\sum_{i=1}^{n-1}\varphi_i(x_1, \ldots, x_i)z_i, \tag{8}$$

where $\alpha_{i-1}(z_1, \ldots, z_{i-1}, \hat{\theta}) = -C_{i-1}z_{i-1} - z_{i-2} - \varphi_{i-1}^\top(x_1, \ldots, x_{i-1})\hat{\theta} + D^{v}\alpha_{i-2}(z_1, \ldots, z_{i-2}, \hat{\theta}), i = 3, \ldots, n, \alpha_1(z_1, \hat{\theta}) = -C_1 z_1 - \varphi_1^\top(x_1)\hat{\theta}$, $C_1, \ldots, C_n > 0$ are constants. The adaptive parameter $\hat{\theta}$ is updated by (8) and $\Gamma = diag[p_1, \ldots, p_n] > 0$ is the gain matrix of the adaptive law.

Proof: We have the following steps.

Step 1. Let $z_1 = x$ and x_2 viewed as the virtual control, the error $z_2 = x_2 - \alpha_1(z_1, \hat{\theta})$, we have

$$D^{v}z_1 = z_2 + \alpha_1(z_1, \hat{\theta}) + \varphi_1^\top(x_1)\theta.$$

Note $\tilde{\theta} = \theta - \hat{\theta}$, the 1st fractional Lyapunov function $V_1(z_1, \hat{\theta}) = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}\Gamma^{-1}\tilde{\theta}$, we have

$$D^{v}V_1 \leq z_1[z_2 + \alpha_1(z_1, \hat{\theta}) + \varphi_1^\top(x_1)\hat{\theta}] + \tilde{\theta}^\top(\varphi_1(x_1)z_1 - \Gamma^{-1}D^{v}\hat{\theta})$$

If choose $\alpha_1(z_1, \hat{\theta}) = -C_1 z_1 - \varphi_1^\top(x_1)\hat{\theta}$, z_2 and $\tilde{\theta}$ are to be governed to zeros. Thus we have

$$D^{v}V_1 \leq -C_1 z_1^2 + z_1 z_2 + \tilde{\theta}^\top(\varphi_1(x_1)z_1 - \Gamma^{-1}D^{v}\hat{\theta}).$$

Step 2. Let the error $z_3 = x_3 - \alpha_2(z_1, z_2, \hat{\theta})$, we have

$$D^{v}z_2 = z_3 + \alpha_2(z_1, z_2, \hat{\theta}) + \varphi_2^\top(x_1, x_2)\theta - D^{v}\alpha_1(z_1, \hat{\theta}).$$

Let 2nd fractional Lyapunov function $V_2(z_1, z_2, \hat{\theta}) = V_1 + \frac{1}{2}z_2^2$, we have

$$\begin{aligned} D^{v}V_2 \leq & -C_1 z_1^2 + z_1 z_2 + \tilde{\theta}^\top(\sum_{i=1}^{2}\varphi_i(x_1, \ldots, x_i)z_i - \Gamma^{-1}D^{v}\hat{\theta}) \\ & + z_2[z_3 + \alpha_2(z_1, z_2, \hat{\theta}) + \varphi_2^\top(x_1 . x_2)\hat{\theta} - D^{v}\alpha_1(z_1, \hat{\theta})]. \end{aligned}$$

If choose $\alpha_2(z_1, z_2, \hat{\theta}) = -C_2 z_2 - z_1 - \varphi_2^\top(x_1, x_2)\hat{\theta} + D^{v}\alpha_1(z_1, \hat{\theta})$, z_3 and $\tilde{\theta}$ are to be governed to zeros. Thus we have

$$D^{v}V_2 \leq -\sum_{i=1}^{2}C_i z_i^2 + z_2 z_3 + \tilde{\theta}^\top(\sum_{i=1}^{2}\varphi_i(x_1, \ldots, x_i)z_i - \Gamma^{-1}D^{v}\hat{\theta}).$$

Step 3. Let the error $z_4 = x_4 - \alpha_3(z_1, z_2, z_3, \hat{\theta})$, we have

$$\begin{aligned} D^{v}z_3 = & z_4 + \alpha_3(z_1, z_2, z_3, \hat{\theta}) + \varphi_3^\top(x_1, x_2, x_3)\theta \\ & - D^{v}\alpha_2(z_1, z_2, \hat{\theta}). \end{aligned}$$

Let 3rd fractional Lyapunov function $V_3(z_1,z_2,z_3,\hat{\theta}) = V_2 + \frac{1}{2}z_3^2$, we have

$$D^\nu V_3 \le -\sum_{i=1}^{2} C_i z_i^2 + z_2 z_3 + \tilde{\theta}^\top (\sum_{i=1}^{3} \varphi_i(x_1,\ldots,x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta})$$
$$+ z_3[z_4 + \alpha_3(z_1,z_2,z_3,\hat{\theta}) + \varphi_3^\top(x_1.x_2,x_3)\hat{\theta} - D^\nu \alpha_2(z_1,z_2,\hat{\theta})].$$

If choose $\alpha_3(z_1,z_2,z_3,\hat{\theta}) = -C_3 z_3 - z_2 - \varphi_3^\top(x_1,x_2,x_3)\hat{\theta} + D^\nu \alpha_2(z_1,z_2,\hat{\theta})$, z_4 and $\tilde{\theta}$ are to be governed to zeros. Thus we have

$$D^\nu V_3 \le -\sum_{i=1}^{3} C_i z_i^2 + z_3 z_4 + \tilde{\theta}^\top (\sum_{i=1}^{3} \varphi_i(x_1,\ldots,x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta}).$$

Step $n-1$. Let the error $z_n = x_n - \alpha_{n-1}(z_1,\ldots,z_{n-1},\hat{\theta})$, we have

$$D^\nu z_{n-1} = z_n + \alpha_{n-1}(z_1,\ldots,z_{n-1},\hat{\theta}) + \varphi_{n-1}^\top(x_1,\ldots,x_{n-1})\theta$$
$$- D^\nu \alpha_{n-2}(z_1,\ldots,z_{n-2},\hat{\theta}).$$

Let $n-1$-th fractional Lyapunov function $V_{n-1}(z_1,\ldots,z_{n-1},\hat{\theta}) = V_{n-1} + \frac{1}{2}z_{n-1}^2$, we have

$$D^\nu V_{n-1} \le -\sum_{i=1}^{n-2} C_i z_i^2 + \tilde{\theta}^\top (\sum_{i=1}^{n-1} \varphi_i(x_1,\ldots,x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta})$$
$$+ z_{n-1}[z_n + z_{n-2} + \alpha_{n-1}(z_1,\ldots,z_{n-1},\hat{\theta})$$
$$+ \varphi_{n-1}^\top(x_1,\ldots,x_{n-1})\hat{\theta} - D^\nu \alpha_{n-2}(z_1,\ldots,z_{n-2},\hat{\theta})].$$

If choose $\alpha_{n-1}(z_1,\ldots,z_{n-1},\hat{\theta}) = -C_{n-1}z_{n-1} - z_{n-2} - \varphi_{n-1}^\top(x_1,\ldots,x_{n-1})\hat{\theta} + D^\nu \alpha_{n-2}(z_1,\ldots,z_{n-2},\hat{\theta})$, z_n and $\tilde{\theta}$ are to be governed to zeros. Thus we have

$$D^\nu V_{n-1} \le -\sum_{i=1}^{n-1} C_i z_i^2 + z_{n-1} z_n$$
$$+ \tilde{\theta}^\top (\sum_{i=1}^{n-1} \varphi_i(x_1,\ldots,x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta}).$$

Step n. The last equation can be transformed into

$$D^\nu z_n = \beta(x) u + \varphi_n^\top(x)\theta - D^\nu \alpha_{n-1}(z_1,\ldots,z_{n-1},\hat{\theta}).$$

Let the acflf (6), we have

$$D^\nu V_a \le -\sum_{i=1}^{n-1} C_i z_i^2 + \tilde{\theta}^\top (\sum_{i=1}^{n-1} \varphi_i(x_1,\ldots,x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta})$$
$$+ z_n[\beta(x)u + z_{n-1} + \varphi_n^\top(x)\hat{\theta} - D^\nu \alpha_{n-1}(z_1,\ldots,z_{n-1},\hat{\theta})].$$

One control and the adaptive law can be chosen by (7) and (8) respectively. Thus we have

$$D^\nu V_a \le -\sum_{i=1}^{n} C_i z_i^2.$$

Consider two cases (i) and (ii):

(i) when $z \neq 0$, we know $D^\nu V_a < 0$. There exists a K-class function γ_1 such that $D^\nu V_a \le -\gamma_1(\|\bar{z}\|), \bar{z} = [z_1,\ldots,z_n,\tilde{\theta}^\top]^\top$.

(ii) when $z = 0$, we know $D^\nu V_a \le 0$. According to the fractional comparison principle [26], we know that

$$D^\nu V_a \le D^\nu C \Longrightarrow V_a \le C,$$

where $C = V_a(t=0)$ is a positive constant.

With respect to Theorem 2.1, for the case (i), the closed-loop system is asymptotically Mittag-Leffler stable on the region Λ. Furthermore, on $\mathbb{R}^{m+n}\backslash\Lambda$, when $V_a(t=0) = 0$, the parameter estimates are asymptotically Mittag-Leffler stable; otherwise, they are bounded.

Therefore, the acflf (6) holds on Λ. So far, this proof is completed. ■

Theorems 3.1 and 3.2 show how to cope with unknown parameters via fractional-order backstepping, which may result in adaptive fractional-order feedback control laws.

IV. NUMERICAL SIMULATION

The usefulness of the proposed method is validated in this section. The gyroscope is a widely used dynamical system and its fractional-order nonlinear model attracts a recent attention [19]. The Grünwald-Letnikov difference [4] is used to simulate the fractional-order nonlinear systems. In the simulation, we abandon the short memory principle for improving numerical accuracy. The time step is $h = 0.0001$.

Example 4.1: The fractional-order gyroscope with control u can be represented by

$$\begin{cases} D^\nu x_1 = x_2 \\ D^\nu x_2 = -p(t)x_1 - c_1 x_2 - c_2 x_2^3 + q(t)x_1^3 + u \end{cases}$$

where $p(t) = \frac{\alpha^2}{4} - f\sin(\omega t), q(t) = \frac{\alpha^2}{12} - \frac{\beta}{6} - \frac{f\sin(\omega t)}{6}, \alpha^2 = 100, \beta = 1, \omega = 25, f = 35.5, \nu = 0.7$ and c_1, c_2 are viewed as unknown constants.

Let $z_1 = x_1$, view x_2 as the virtual control and $z_2 = x_2 - \alpha_1$, we have $D^\nu z_1 = z_2 + \alpha_1(z_1,\hat{c_1},\hat{c_2})$. Denote $\tilde{c_1} = c_1 - \hat{c_1}, \tilde{c_2} = c_2 - \hat{c_2}$. Let the 1st fractional Lyapunov function $V_1 = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}\tilde{c_1}^2 + \frac{1}{2\rho}\tilde{c_2}^2$.

If choose $\alpha_1(z_1,\hat{c_1},\hat{c_2}) = -K_1 x_1, K_1 > 0$, we have

$$D^\nu V_1 \le -K_1 z_1^2 + z_1 z_2 - \frac{1}{\gamma}\tilde{c_1} D^\nu \hat{c_1} - \frac{1}{\rho}\tilde{c_2} D^\nu \hat{c_2}.$$

Step 2. With $D^\nu z_2 = -p(t)x_1 - c_1 x_2 - c_2 x_2^3 + q(t)x_1^3 + u - D^\nu \alpha_1$, let the candidate acflf $V_2 = V_1 + \frac{1}{2}z_2^2$.

The adaptive control law can be chosen by

$$u = (-K_1K_2 + p(t) - 1)x_1 + (-K_2 + \hat{c_1} - K_1)x_2 + \hat{c_2}x_2^3$$
$$- q(t)x_1^3, K_2 > 0,$$
$$D^\nu \hat{c}_1 = -\gamma x_2 z_2, D^\nu \hat{c}_2 = \rho x_2^3 z_2.$$

Hence, we have $D^{\nu}V_2 \leq -K_1 z_1^2 - K_2 z_2^2$.

In the simulation, $K_1 = 2, K_2 = 3, \rho = \gamma = 1$. The initial state is $(2, -2)$ and the initial parameter estimate $(0, 0)$. The unknown parameters are set to $c_1 = 0.5, c_2 = 0.05$. The state trajectories of the controlled system are shown in Fig. 1. By applying the control input, the system converges to the equilibrium quickly. The control input is shown in Fig. 2. The parameter estimates are shown in Fig. 3. It is verified that the adaptive fractional-order backstepping is feasible for real fractional-order systems.

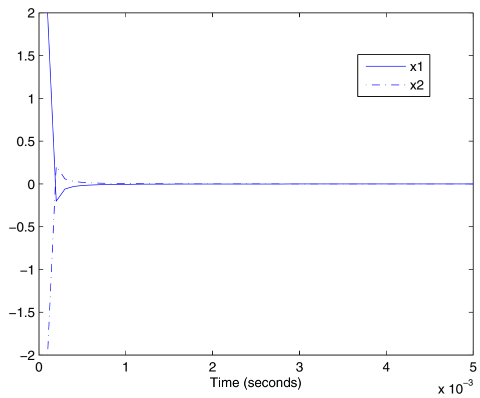

Fig. 1. The states

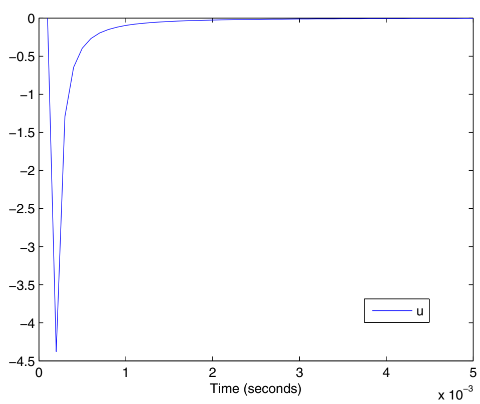

Fig. 2. The control input

V. CONCLUSIONS AND FUTURE WORKS

This paper concerns the Mittag-Leffler stabilization problem. A general framework of Lyapunov-like based design is defined via adaptive control fractional Lyapunov function.

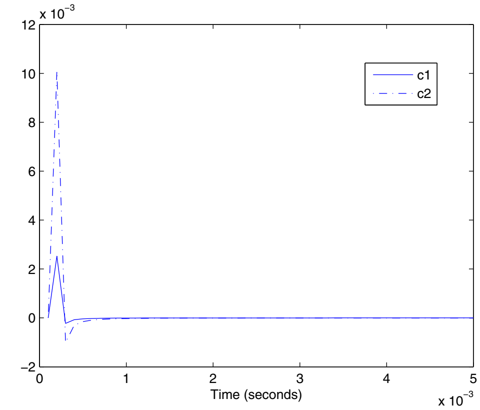

Fig. 3. The parameter estimates

Within this framework, the adaptive fractional-order backstepping technique is proposed. The analytic forms of adaptive feedback control laws are designed via this technique. The proposed development is verified finally.

There are many promising directions for future works. Uncertain fractional-order nonlinear systems need to be investigated further with unmatched disturbances. Some applications are still meaningful to the success of fractional-order backstepping technique. Furthermore, the relationship between Lyapunov function and fractional Lyapunov function should be bridged.

VI. ACKNOWLEDGMENTS

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